Playing Large Games

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Defn: Given 2-player game $R, C \in (0,1)^{n \times n}$, $\varepsilon > 0$, strategy profile (x^*, y^*), x^* is an ε-best-response to y^* if:

$$x^{*T} R y^* \geq (R y^*)_i - \varepsilon \quad \forall i \in [n]$$

Claim: Given 2-player game $R, C \in (0,1)^{n \times n}$, $\varepsilon > 0$ strategy profile (x^*, y^*) s.t.

$\forall i, i' \in [n]$, $x_i^* > 0 \Rightarrow (Ry^*)_i \geq (Ry^*)_{i'}$

then x^* is an ε-BR to y^*

Defn: Given 2-player game $R, C \in (0,1)^{n \times n}$, $\varepsilon > 0$, strategy profile (x^*, y^*) is an ε-NE if x^* is an ε-BR to y^*, & y^* is an ε-BR to x^*

i.e., $x^{*T} R y^* \geq (R y^*)_i - \varepsilon \quad \forall i \in [n]$

& $y^{*T} C x^* \geq (C x^*)_j - \varepsilon \quad \forall j \in [n]$

Theorem: Given 2-player game $R, C \in (0,1)^{n \times n}$, $\varepsilon > 0$, $\exists$ an ε-NE $\hat{x}, \hat{y}$ s.t. $|\text{supp}(\hat{x})| \leq 12 \ln n / \varepsilon^2$ & $|\text{supp}(\hat{y})| \leq 12 \ln n / \varepsilon^2$

Corollary: Given 2-player game $R, C \in (0,1)^{n \times n}$, $\varepsilon > 0$, an ε-NE can be computed in time $O(n^{24 \ln n/\varepsilon^2} \cdot \text{poly}(n, R, C))$

(guess $\text{supp}(\hat{x})$, $\text{supp}(\hat{y})$. $\binom{n}{12 \ln n/\varepsilon^2}$ choices for each)

Will use Hoeffding's inequality:

Let $X_1, \dots, X_k$ be independent r.v.'s in $[0,1]$. Let $\mu = \frac{1}{k} \sum_i X_i$. Then

$$\Pr\left[\left| \frac{1}{k} \sum_i X_i - \mathbb{E}[\mu] \right| > \varepsilon \right] \leq 2 e^{-2k\varepsilon^2}$$

Proof of Theorem: Let (x^*, y^*) be a NE. Let $k = 12 \ln n / \varepsilon^2$.

Let A be a multiset of k pure strategies, sampled independently from x^*, & B be a multiset of k pure strategies, sampled independently from y^*.

Let $\hat{x}, \hat{y}$ be the emprical distribution of pure strategies in x^*. i.e., if pure strategy i appears r_i times in A,

$$\hat{x}_i = r_i / k.$$

We will show that $\hat{x}, \hat{y}$ is the required ε-NE.

Note that: $|\text{supp}(\hat{x})|, |\text{supp}(\hat{y})| \leq k$

$\text{supp}(\hat{x}) \subseteq \text{supp}(x^*)$

$\text{supp}(\hat{y}) \subseteq \text{supp}(y^*)$

We need to show: $\hat{x}$ is ε-BR to $\hat{y}$, & $\hat{y}$ is ε-BR to $\hat{x}$.

Will show:

$$\forall i, \quad \hat{x}_i > 0 \Rightarrow (R\hat{y})_i \geq (R\hat{y})_{i'} - \varepsilon$$

Fix $i \in [n]$. Consider r.v. Y s.t.

$$Y = R_{ij} \quad \text{w.p.} \quad y_j^*$$

For k samples $Y^1 \dots Y^k$,

$$\Pr\left[\left| \frac{1}{k} \sum_{r=1}^{k} Y^r - (Ry^*)_i \right| > \varepsilon/2 \right] \leq 2e^{-k\varepsilon^2/2}$$

or $$\Pr\left[|(R\hat{y})_i - (Ry^*)_i| > \varepsilon/2 \right] \leq 2e^{-k\varepsilon^2/2}$$

Thus w.p. $1 - 2n e^{-k\varepsilon^2/2}$, $\forall i$, $|(R\hat{y})_i - (Ry^*)_i| \leq \varepsilon/2$

Now consider $i: \hat{x}_i > 0 \Rightarrow x_i^* > 0$

$\Rightarrow (Ry^*)_i \geq (Ry^*)_{i'}$

Thus w.p. $1 - 2n e^{-k\varepsilon^2/2}$, $\hat{x}_i > 0 \Rightarrow (R\hat{y})_i \geq (Ry^*)_i - \varepsilon/2$

$\geq (Ry^*)_{i'} - \varepsilon/2$

$\geq (R\hat{y})_{i'} - \varepsilon$

Similarly, w.p. $1 - 4n e^{-k\varepsilon^2/2}$, $\hat{y}_j > 0 \Rightarrow (C\hat{x})_j \geq (C\hat{x})_{j'} - \varepsilon$.

Hence, w.p. $1 - 4n e^{-k\varepsilon^2/2}$, $(\hat{x}, \hat{y})$ is an ε-NE.

now $1 - 4n e^{-k\varepsilon^2/2} = 1 - 4n \exp\left(- \frac{12 \ln n}{\varepsilon^2} \cdot \frac{\varepsilon^2}{2} \right)$

$= 1 - 4n \cdot \frac{1}{n^6} > 0$

Hence, $\exists\ \hat{x}, \hat{y}$ s.t. $|\text{supp}(\hat{x})|, |\text{supp}(\hat{y})| \leq k$ & $\hat{x}, \hat{y}$ is an ε-NE.

(This is known as the probabilistic method). ∎

Such a strategy is called a k-uniform strategy, where $\forall i$, $\hat{x}_i = \frac{\lambda}{k}$, $\lambda \in \mathbb{Z}_+$

Can show:

1. $|\hat{x}^T R \hat{y} - x^{*T} R y^*| \leq \varepsilon$
   $|\hat{y}^T C \hat{x} - y^{*T} C x^*| \leq \varepsilon$

2. For m players w/ n pure strategies each, there $\exists$ k-uniform strategies $\hat{s}_1, \dots, \hat{s}_m$ for the players s.t. $|\text{supp}(\hat{s}_i)| \leq k$, & $\hat{s} = (\hat{s}_1, \dots, \hat{s}_m)$ is an ε-NE for $\varepsilon = \frac{3m^2 \ln m^2 n}{\varepsilon^2}$

k reduced to $\frac{8 \ln mn}{\varepsilon^2}$ in later work.